

## Appendix A

### Discrete Fourier Transform

The Discrete Fourier Transform (DFT) is a method for finding the frequency components of a signal. (It is named after Jean Baptiste Joseph Fourier, an early 19<sup>th</sup> century French mathematician.) The human ear does something very similar using the cochlea. The cochlea is a mechanical filter whereas the DFT is a mathematical filter. Both serve a similar purpose in speech recognition, that of transforming incoming sound into its frequency components.

Figure A.1 is a spreadsheet example of the DFT to extract the frequency components of a signal  $S(t)$ . In effect, the sine and cosine of various frequencies are correlated against the signal  $S(t)$ . Here we check only three frequencies since we know apriori the frequencies in  $S(t)$ . Normally, for the 32 sample signal shown, 16 frequencies would be checked (16 sine and/or cosine values = 32 coefficients). These are  $\cos(n\omega t)$ ,  $n = 0, \dots, 16$ , and  $\sin(n\omega t)$ ,  $n = 1, \dots, 15$ . This insures that all of the energy in the time domain signal is accounted for in the frequency domain representation.

The important point to note here from the spreadsheet is that each frequency component may be found independently of the others. That is, the determination of the magnitude of  $\sin(n\omega t_j)$  or  $\cos(n\omega t_j)$  is found by calculating

$$A_n = \frac{\sum_{j=0}^{j=M-1} \cos(n \cdot \omega \cdot t_j) \cdot S(t_j)}{\sum_{j=0}^{j=M-1} \cos^2(n \cdot \omega \cdot t_j)} \quad \text{and} \quad B_n = \frac{\sum_{j=0}^{j=M-1} \sin(n \cdot \omega \cdot t_j) \cdot S(t_j)}{\sum_{j=0}^{j=M-1} \sin^2(n \cdot \omega \cdot t_j)} \quad (1)$$

where  $M = \text{Number of Samples}$ .

All this begs the question, how did this formula come about. Why would anyone think to multiply two sequences together (say  $\sin(n\omega t_j)$  and  $S(t_j)$ ) and then sum the individual products? The answer lies in the Method of Least Squares, one of the most powerful concepts in applied mathematics. It underlies Kalman filters, pattern recognition, machine learning, and, of course, regression analysis, among others. See the Wikipedia entry [http://en.wikipedia.org/wiki/Least\\_squares](http://en.wikipedia.org/wiki/Least_squares).

The German mathematician Karl Friederich Gauss is credited with the discovery of the Method of Least Squares. Independently, the French mathematician Adrien-Marie Legendre discovered the method also. Much to the chagrin of Gauss, Legendre actually published before Gauss. This led to a feud between these two great mathematicians.

Briefly, the method finds the coefficients  $c_n$  in a mathematical model (e.g.,  $c_0 + c_1 \cdot x + c_2 \cdot x^2$ ) that minimizes the sum of the squares of the difference between a given data set (sequence of numbers) and the model. That is, for our simple polynomial example:

$$\sum_{j=0}^{j=M-1} (S_j - (c_0 + c_1 \cdot x_j + c_2 \cdot x_j^2))^2 = \text{minimum} \quad (2)$$

Using differential calculus (differentiating with respect to  $c_0$ ,  $c_1$ ,  $c_2$  and setting each result equal to zero), it is straightforward to show that this minimization leads to three simultaneous linear equations in  $c_0$ ,  $c_1$ ,  $c_2$ :

$$c_0 \cdot \sum_{j=0}^{j=M-1} 1 + c_1 \cdot \sum_{j=0}^{j=M-1} x_j \cdot 1 + c_2 \cdot \sum_{j=0}^{j=M-1} x_j^2 \cdot 1 = \sum_{j=0}^{j=M-1} S_j \cdot 1 \quad (3a)$$

$$c_0 \cdot \sum_{j=0}^{j=M-1} x_j + c_1 \cdot \sum_{j=0}^{j=M-1} x_j^2 + c_2 \cdot \sum_{j=0}^{j=M-1} x_j^3 = \sum_{j=0}^{j=M-1} S_j \cdot x_j \quad (3b)$$

$$c_0 \cdot \sum_{j=0}^{j=M-1} x_j^2 + c_1 \cdot \sum_{j=0}^{j=M-1} x_j^3 + c_2 \cdot \sum_{j=0}^{j=M-1} x_j^4 = \sum_{j=0}^{j=M-1} S_j \cdot x_j^2 \quad (3c)$$

While there are many methods for solving simultaneous linear equations, if the mathematical model consists of sines and cosines whose frequency corresponds to integer multiple wavelengths of the sample segment interval (the DFT meets this requirement), then each equation reduces to:

$$A_n \cdot \sum_{j=0}^{j=M-1} \cos^2(n \cdot \omega \cdot t_j) = \sum_{j=0}^{j=M-1} \cos(n \cdot \omega \cdot t_j) \cdot S(t_j) \quad (4a)$$

$$B_n \cdot \sum_{j=0}^{j=M-1} \sin^2(n \cdot \omega \cdot t_j) = \sum_{j=0}^{j=M-1} \sin(n \cdot \omega \cdot t_j) \cdot S(t_j) \quad (4b)$$

which may be readily solved for  $A_n$  and  $B_n$ , as shown on the previous page.

If all of this just seems like so many Greek characters, the spreadsheet of Figure A.1 is an illustration of this process. Column A is the time in milliseconds, column B is the angle  $\omega t_j$ . The angular frequency  $\omega$  is chosen such that  $\omega \Delta T = 2\pi$ .  $\Delta T$  is the sample segment interval.

Columns C through H are the numerical values of the sines and cosines of  $n\omega t_j$ ,  $n = 1, 2, 3$  at the times  $t_j = 0, 1, \dots, 31$ . Similarly, column I contains the numerical values of the assumed signal

$$S(t_j) = 2 \cdot \sin(\omega t_j) - 3 \cdot \cos(2\omega t_j) + 1 \cdot \sin(3\omega t_j).$$

If we sum the products of any two pair of columns C through I, we obtained the 6 x 7 array at the bottom of the page. The left 6 x 6 array confirms that the sum of products of columns C through H that

are paired with themselves are non-zero. This is another way of illustrating the orthogonality of the sine and cosine functions in the interval  $\Delta T$  with  $\omega = 2\pi / \Delta T$ . The summations on the left side of equations (4a) and (4b) are the non-zero result ( $= M/2 = 16$  in the spreadsheet).

The seventh column is the sum of the products of columns C through H paired with column I. This is the right side of equations (4a) and (4b).

Thus, to find the  $A_n$  and  $B_n$ , we need only divide the results in the last column by the non-zero result in the same row of the 6 x 6 array. Thus,

$$B_1 = 32/16 = 2, \quad A_1 = 0, \quad B_2 = 0, \quad A_2 = -48/16 = -3, \quad B_3 = 16/16 = 1, \quad \text{and } A_3 = 0.$$

The coefficients recovered are identical to the coefficients in our assumed signal  $S(t)$ . The lower right corner spreadsheet calculation obtains the same result.

Interestingly, if we calculate the power in  $S(t)$  and in the frequency components, we find they are equal.

First, the frequency components:

$$\begin{aligned} P(A_n, B_n) &= \frac{1}{2} \sum_{n=1}^3 (A_n^2 + B_n^2) \\ &= \{2^2 + (-3)^2 + 1^2\} / 2 = 7 \end{aligned}$$

The factor 1/2 is required because the components are peak values, not RMS.

The power in the signal  $S(t_j)$  is:

$$P(S(t)) = \frac{1}{M} \sum_{j=0}^{j=M-1} S(t_j)^2 = 7 \quad (\text{From the spreadsheet})$$

This calculation is from the spreadsheet. The fact that  $P(A_n, B_n) = P(S(t))$  is known as Parseval's Theorem. In effect, no energy is lost or gained transforming from the time domain to the frequency domain.

As a final comment, the Fast Fourier Transform produces the same result, but takes advantage of certain redundancies in the calculation if  $M$  is a power of 2, e.g. 2, 4, 8, 16, 32, 64, etc. See the Wikipedia entry [http://en.wikipedia.org/wiki/Fast\\_Fourier\\_transform](http://en.wikipedia.org/wiki/Fast_Fourier_transform).

The functioning spreadsheet may be found on the CD-ROM.

## A Simple Non-Rigorous Example of Frequency Component Determination

$$\pi = 3.14152$$

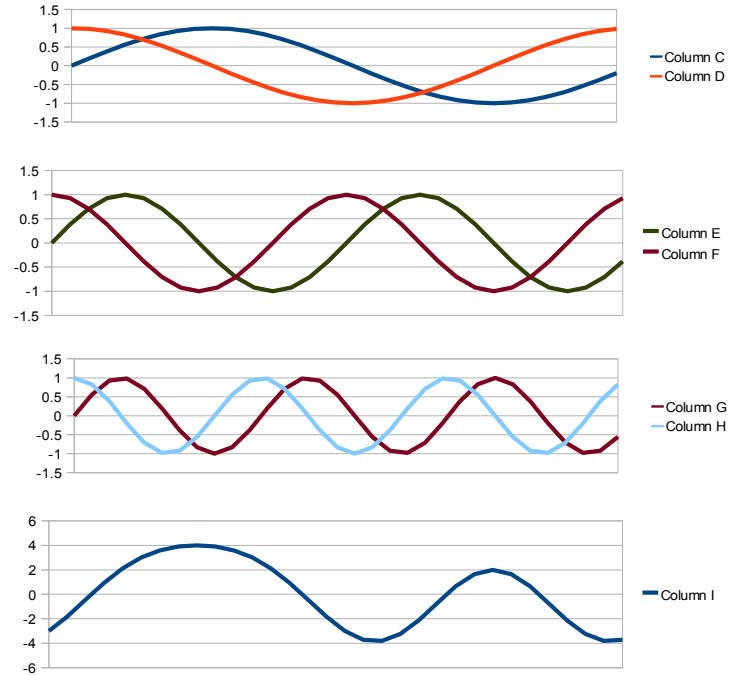
$$\omega = 2\pi / \Delta T = 0.19635$$

$$\text{Samples} = 32$$

$$\Delta T = 32$$

$$S(t) = 2\sin(\omega t) - 3\cos(2\omega t) + \sin(3\omega t)$$

A	B	C	D	E	F	G	H	I
t	$\omega t$	$\sin(\omega t)$	$\cos(\omega t)$	$\sin(2\omega t)$	$\cos(2\omega t)$	$\sin(3\omega t)$	$\cos(3\omega t)$	$S(t)$
0	0	0	1	0	1	0	1	-3
1	0.2	0.2	0.98	0.38	0.92	0.56	0.83	-1.83
2	0.39	0.38	0.92	0.71	0.71	0.92	0.38	-0.43
3	0.59	0.56	0.83	0.92	0.38	0.98	-0.2	0.94
4	0.79	0.71	0.71	1	0	0.71	-0.71	2.12
5	0.98	0.83	0.56	0.92	-0.38	0.2	-0.98	3.01
6	1.18	0.92	0.38	0.71	-0.71	-0.38	-0.92	3.59
7	1.37	0.98	0.2	0.38	-0.92	-0.83	-0.56	3.9
8	1.57	1	0	0	-1	-1	0	4
9	1.77	0.98	-0.2	-0.38	-0.92	-0.83	0.56	3.9
10	1.96	0.92	-0.38	-0.71	-0.71	-0.38	0.92	3.59
11	2.16	0.83	-0.56	-0.92	-0.38	0.19	0.98	3.01
12	2.36	0.71	-0.71	-1	0	0.71	0.71	2.12
13	2.55	0.56	-0.83	-0.92	0.38	0.98	0.2	0.94
14	2.75	0.38	-0.92	-0.71	0.71	0.92	-0.38	-0.43
15	2.95	0.2	-0.98	-0.38	0.92	0.56	-0.83	-1.83
16	3.14	0	-1	0	1	0	-1	-3
17	3.34	-0.2	-0.98	0.38	0.92	-0.56	-0.83	-3.72
18	3.53	-0.38	-0.92	0.71	0.71	-0.92	-0.38	-3.81
19	3.73	-0.56	-0.83	0.92	0.38	-0.98	0.19	-3.24
20	3.93	-0.71	-0.71	1	0	-0.71	0.71	-2.12
21	4.12	-0.83	-0.56	0.92	-0.38	-0.2	0.98	-0.71
22	4.32	-0.92	-0.38	0.71	-0.71	0.38	0.92	0.66
23	4.52	-0.98	-0.2	0.38	-0.92	0.83	0.56	1.64
24	4.71	-1	0	0	-1	1	0	2
25	4.91	-0.98	0.19	-0.38	-0.92	0.83	-0.56	1.64
26	5.1	-0.92	0.38	-0.71	-0.71	0.38	-0.92	0.66
27	5.3	-0.83	0.56	-0.92	-0.38	-0.19	-0.98	-0.71
28	5.5	-0.71	0.71	-1	0	-0.71	-0.71	-2.12
29	5.69	-0.56	0.83	-0.92	0.38	-0.98	-0.2	-3.24
30	5.89	-0.38	0.92	-0.71	0.71	-0.92	0.38	-3.81
31	6.09	-0.2	0.98	-0.38	0.92	-0.56	0.83	-3.72



	$\sin(\omega t)$	$\cos(\omega t)$	$\sin(2\omega t)$	$\cos(2\omega t)$	$\sin(3\omega t)$	$\cos(3\omega t)$	$S(t)$
$\sin(\omega t)$	16	0	0	0	0	0	32
$\cos(\omega t)$	0	16	0	0	0	0	0
$\sin(2\omega t)$	0	0	16	0	0	0	0
$\cos(2\omega t)$	0	0	0	16	0	0	-48
$\sin(3\omega t)$	0	0	0	0	16	0	16
$\cos(3\omega t)$	0	0	0	0	0	16	0

Above entries are the sum of products of the row and column functions, e.g.,  $\sum[\cos(2\omega t) \sin(3\omega t)]$ .

The matrix at left implies:

- $\sum[\sin(n\omega t) \sin(m\omega t)] = 16$ , if  $m = n$ , = 0 otherwise
- $\sum[\cos(n\omega t) \cos(m\omega t)] = 16$ , if  $m = n$ , = 0 otherwise
- $\sum[\sin(n\omega t) \cos(m\omega t)] = 0$ , all  $m, n$

The above property of the sines and cosines is orthogonality. We may use this property to easily find the frequency components of a signal.

If  $S(t) = 2\sin(\omega t) - 3\cos(2\omega t) + 1\sin(3\omega t)$  (See Column I,  $S(t)$ ), then computing

$$\frac{\sum[\sin(n\omega t) S(t)]}{\sum[\sin(n\omega t) \sin(n\omega t)]} \text{ and } \frac{\sum[\cos(n\omega t) S(t)]}{\sum[\cos(n\omega t) \cos(n\omega t)]}$$

for  $S(t)$  and  $n = 1, 2, 3$ , (using the precomputed values from the array at left), yields the following:

$\sin(\omega t)$	$\cos(\omega t)$	$\sin(2\omega t)$	$\cos(2\omega t)$	$\sin(3\omega t)$	$\cos(3\omega t)$
2	0	0	-3	1	0

Parseval's Theorem: Power in signal  $S(t) = \{\sum S(t)^2\} / \text{Samples} = 7 = \text{Power in frequency components} = \{2^2 + (-3)^2 + 1^2\}$

